Nonextensive foundation of Lévy distributions

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A deep connection between the ubiquity of Lévy distributions in nature and the nonextensive thermal statistics introduced a decade ago has been established recently [Tsallis *et al.*, Phys. Rev. Lett. **75**, 3589 (1995)], by using *unnormalized* q-expectation values. It has just been argued on physical grounds that *normalized* q-expectation values should be used instead. We revisit, within this more appropriate scheme, the Lévy problem and verify that the relevant analytic results become sensibly simplified, whereas the basic physics remains unchanged. [S1063-651X(99)14008-X]

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Gaussian distributions are ubiquitous in nature, and are known to be intimately related to *normal* (Brownian) diffusion. However, it is by now well established ([1], and references therein) that another large variety of stochastic phenomena [*anomalous* (super) diffusion] in physical and even biological and socioeconomical sciences is controlled by Lévy distributions. A basic question is to understand what is the thermostatistical foundation of this fact. Such an answer is well known for normal diffusion. Indeed, it is based on two pillars, namely, the Boltzmann-Gibbs (BG) entropy and the standard central-limit theorem. What would be the analogous basis for Lévy superdiffusion? Montroll and co-workers have specifically addressed this interesting question (see [2], and references therein). They showed that if the BG entropy is used, the auxiliary constraint to be imposed in order to obtain Lévy distributions is unacceptable as an *a priori* constraint. Indeed, only a complex *ad hoc* constraint yields Lévy distributions. Naturally, they considered that procedure far from satisfactory. A few years ago, this puzzle was essentially solved [3,4] within the framework of a generalized thermostatistics, which uses a *nonextensive* entropy. Before further details, let us address the case of the standard, normal diffusion, in terms of a variational principle. The BG entropy associated with one particle diffusing along the x axis (starting at $t=0$, at the origin $x=0$) is given by

$$S_1[p] = -k\int_{-\infty}^{\infty} dx\, p(x)\ln[\sigma p(x)] \quad (k>0) \tag{1}$$

(the subindex 1 will soon become transparent; $\sigma>0$ is a characteristic length) with

$$\int_{-\infty}^{\infty} dx\, p(x) = 1. \tag{2}$$

The simplest additional constraint in order to catch the essentials of normal (unbiased, i.e., symmetric) diffusion is given by

$$\langle x^2\rangle_1 \equiv \int_{-\infty}^{\infty} dx\, x^2\, p(x) = \sigma^2. \tag{3}$$

Using Lagrange parameters, we immediately obtain the *one*-jump distribution, which optimizes $S_1[p]$:

$$p_1(x) = \exp(-\beta x^2)/Z_1 \tag{4}$$

with $Z_1 \equiv \int_{-\infty}^{\infty} dx \exp(-\beta x^2) = (\pi/\beta)^{1/2}$. The substitution of Eq. (4) into Eq. (3) yields $\beta \equiv 1/kT = 1/(2\sigma^2)$. We next want to find the distribution $p_1(x,N)$ associated with the *macroscopic* phenomenon (N jumps). This is given by the N-folded convolution product $p_1(x,N) = p_1(x) * \dots * p_1(x)$ (N times). Replacing Eq. (4) into this product yields

$$p_1(x,N) = \left(\frac{\beta}{\pi N}\right)^{1/2} \exp\left(-\frac{\beta x^2}{N}\right); \tag{5}$$

hence,

$$p_1(x,N) = \frac{1}{N^{1/2}} p_1\left(\frac{x}{N^{1/2}}\right). \tag{6}$$

Finally, it follows that

$$\langle x^2\rangle_1(N) \equiv \int_{-\infty}^{\infty} dx\, x^2\, p_1(x,N) = \frac{1}{2}kTN. \tag{7}$$

Assuming that $N=Dt$, where t is time and D^{-1} is the characteristic time of the problem, we recover the celebrated Einstein 1905 result ($\langle x^2\rangle_1 = DkTt/2$). Due to the standard central-limit theorem, if the one-jump distribution were not that given in Eq. (4), but an arbitrary one with *finite* second moment σ^2, the N-jump distribution would be, in the asymptotic limit $N\to\infty$, exactly the same as obtained above [Eq. (5)]. Summarizing, the thermostatistical foundation of Gaussians in nature is based on the BG entropy and the standard central-limit theorem. Let us now generalize the above beautiful scheme in order to also cover Lévy distributions. To do this, we start by considering the following generalized, nonextensive entropic form [5]:

$$S_q[p(x)] = k\,\frac{1-\int_{-\infty}^{\infty}\frac{dx}{\sigma}[\sigma p(x)]^q}{q-1}. \tag{8}$$

The nonextensive statistical mechanics generated by this entropy has been usefully applied. We can mention two-dimensionallike turbulence in pure-electron plasma [6], cos-

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mology [7], dissipative maps [8], and self-organized criticality [9], among others (see also [10]). Following along Gibbs' path, we wish to optimize S_q with the constraint given in Eq. (2) and also

$$\langle x^2\rangle_q \equiv \frac{\int_{-\infty}^{\infty} dx\, x^2\,[p(x)]^q}{\int_{-\infty}^{\infty} dx\,[p(x)]^q} = \sigma^2. \tag{9}$$

We shall from now refer to the above q-expectation value as the *normalized* one, in contrast to the *unnormalized* one ($\int_{-\infty}^{\infty} dx\, x^2\,[p(x)]^q$) used in [4]. It is this substantial difference that makes the revisiting of this problem necessary. The introduction [11] of *normalized* q-expectation values has been proven to be the correct formulation of the nonextensive statistics (see [11,12]). The optimization of Eq. (8) with constraints (2) and (9) straightforwardly yields

$$p_q(x) = \frac{1}{\sigma}\left[\frac{q-1}{\pi(3-q)}\right]^{1/2} \frac{\Gamma\left(\frac{1}{q-1}\right)}{\Gamma\left(\frac{3-q}{2(q-1)}\right)} \frac{1}{\left[1+\frac{q-1}{3-q}\frac{x^2}{\sigma^2}\right]^{1/(q-1)}} \tag{10}$$

for $q>1$, and, for $q<1$,

$$p_q(x) = \frac{1}{\sigma}\left[\frac{1-q}{\pi(3-q)}\right]^{1/2} \frac{\Gamma\left(\frac{5-3q}{2(1-q)}\right)}{\Gamma\left(\frac{2-q}{1-q}\right)} \left[1-\frac{1-q}{3-q}\frac{x^2}{\sigma^2}\right]^{1/(1-q)} \tag{11}$$

if $|x|<\sigma[(3-q)/(1-q)]^{1/2}$ and zero otherwise. As we see, in the $q<1$ case, there is a cutoff (compact support). Both limits $q\to 1+0$ and $q\to 1-0$ recover the Gaussian solution [Eq. (4)]. The particular cases $q\to-\infty$, $q=2$, and $q\to 3-0$, respectively, correspond to an uniform, Cauchy-Lorentz, and completely flat distribution; no distribution exists for $q\geqslant 3$ because Eq. (2) cannot be satisfied. Finally, we can verify that $q<5/3$ ($q\geqslant 5/3$) implies a *finite* (*infinite*) one-jump second moment $\langle x^2\rangle_1$. See Fig. 1. It is worth mentioning at this point that the functional form of the distributions appearing in Eqs. (10) and (11) has been shown [13] to generate the *exact* solution $[\forall(x,t)]$ of a correlated anomalous diffusion problem. Before continuing, in order to avoid any notation confusion, let us emphasize that we are from now on using the q-expectation values as follows: $\langle A(x)\rangle_1 \equiv \int_{\infty}^{\infty} dx\, A(x) p_q(x)$ and

$$\langle A(x)\rangle_q \equiv \frac{\int_{-\infty}^{\infty} dx\, A(x)[p_q(x)]^q}{\int_{-\infty}^{\infty} dx[p_q(x)]^q},$$

where $A(x)$ is an arbitrary function. Consistently, we have that $\langle A(x)\rangle_1(N) \equiv \int_{-\infty}^{\infty} dx A(x) p_q(x,N)$ and

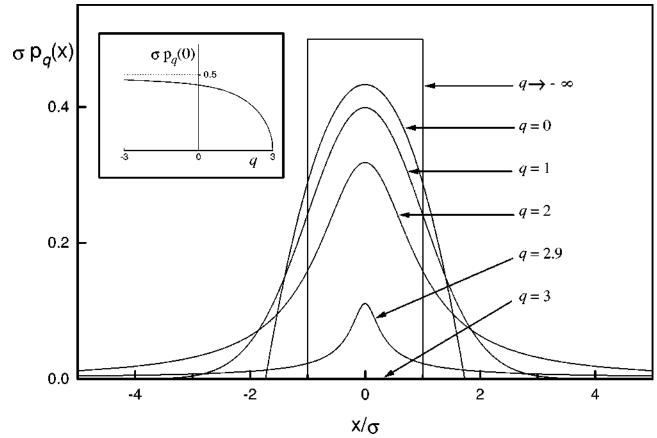

FIG. 1. The one-jump distributions $p_q(x)$ for typical values of q. The $q\to-\infty$ distribution is the uniform one in the interval $[-1,1]$; $q=1$ and $q=2$, respectively, correspond to Gaussian and Lorentzian distributions; the $q\to 3$ distribution is completely flat. For $q<1$, there is a cutoff; for $q>1$, there is a $1/|x|^{2/(q-1)}$ tail at $|x|\gg\sigma$.

$$\langle A(x)\rangle_q(N) \equiv \frac{\int_{-\infty}^{\infty} dx\, A(x)[p_q(x,N)]^q}{\int_{-\infty}^{\infty} dx[p_q(x,N)]^q},$$

where $p_q(x,N)$ is the N-jump distribution, given by the N-folded convolution of $p_q(x)$. Of course, $\langle A(x)\rangle_1(1) = \langle A(x)\rangle_1$ and $\langle A(x)\rangle_q(1) = \langle A(x)\rangle_q$. Let us now continue by addressing the N-jump distribution. Two physical cases have to be distinguished, namely, for q below or above 5/3.

For $q<5/3$, the second moment is finite and given by

$$\langle x^2\rangle_1 = \sigma^2 \frac{3-q}{5-3q} \quad (q<5/3). \tag{12}$$

It is interesting to notice that, although the distributions associated with $q<1$ and $1\leqslant q<5/3$ have different functional expressions [namely, those given in Eqs. (10) and (11), respectively], the above expression is *one and the same*. It is easy to see that the standard central-limit theorem implies that the N-jump distribution is given by a properly scaled Gaussian with the *same* second moment. More specifically, we have, for $N\to\infty$,

$$p_q(x,N) \sim \frac{1}{\sigma}\left[\frac{5-3q}{2\pi(3-q)N}\right]^{1/2} \exp\left(-\frac{5-3q}{2(3-q)N}\frac{x^2}{\sigma^2}\right). \tag{13}$$

We verify that

$$\langle x^2\rangle_1(N) = \langle x^2\rangle_1 N; \tag{14}$$

hence,

$$\langle x^2\rangle_1(N) = \sigma^2 \Delta_q N, \tag{15}$$

where we have introduced a dimensionless diffusion coefficient, namely,

$$\Delta_q \equiv \frac{3-q}{5-3q} \quad (q<5/3). \tag{16}$$

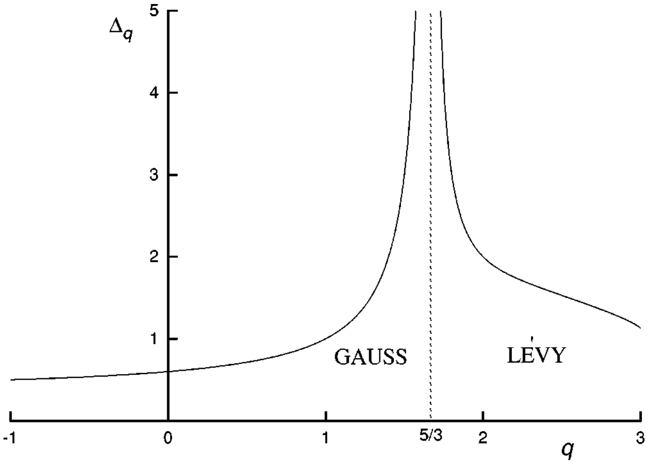

FIG. 2. The q dependence of the dimensionless diffusion coefficient Δ_q [*width* of the properly scaled distribution $p_q(x,N)$ in the $N\to\infty$ limit]. In the limits $q\to 5/3-0$ and $q\to 5/3+0$ we, respectively, have $\Delta_q \sim [4/9]/[(5/3)-q]$ and $\Delta_q \sim [4/(9\pi^{1/2}]/[q-(5/3)]$; also, $\lim_{q\to 3}\Delta_q = 2/\pi^{1/2}$.

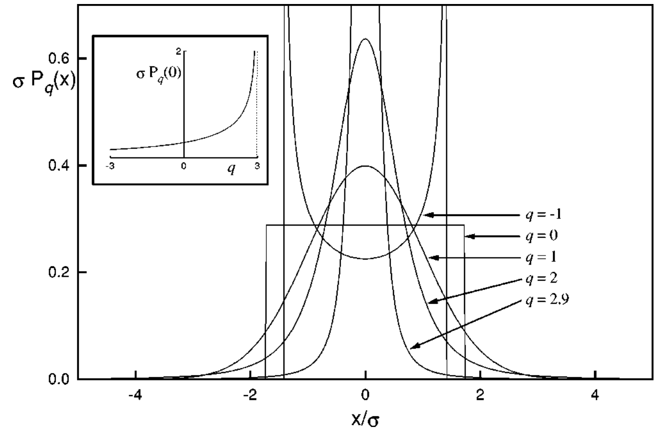

FIG. 3. The one-jump *escort* distributions $P_q(x)$ for typical values of q. The $q=0$ distribution is the uniform one in the interval $[-3^{1/2},3^{1/2}]$; $q=1$ corresponds to the Gaussian; the $q\to 3$ distribution corresponds to Dirac's delta. For $q<1$, there is a cutoff; for $q>1$, there is a $1/|x|^{2q/(q-1)}$ tail at $|x|\gg\sigma$.

(See Fig. 2.) If we expand, for small wave-vector κ, the Fourier-transform $F(\kappa,N)\equiv\int_{-\infty}^{\infty}dx\,\exp[i\kappa x]p_q(x,N)$, we obtain $F(\kappa,N)\sim 1-\kappa^2\langle x^2\rangle_1(N)/2$. Since, from the standard central-limit theorem, we know that $F(\kappa,N) \propto \exp[-N\sigma^2\Delta_q\kappa^2/2]$, we see that Eq. (20) is basically giving the *width* (per unit N) of $p_q(x,N)$. Let us now address the $q>5/3$ case. The second moment [associated with distribution (11)] $\langle x^2\rangle_1$ diverges; hence, what applies is the Lévy-Gnedenko central-limit theorem [14]. In other words, the distribution $p_q(x,N)$ approaches in the $N\to\infty$ limit, a properly scaled Lévy distribution $L_\gamma(x/N^{1/\gamma})$ whose $|x|\to\infty$ asymptotic behavior shares with $p_q(x)$ both the exponent and the coefficient. More precisely, the Fourier transform $F(\kappa,N)$ associated with L_γ is proportional to $\exp[-N\sigma^\gamma\Delta_q|\kappa|^\gamma/2]$, where

$$\gamma=\frac{3-q}{q-1}\quad(5/3<q<3)\tag{17}$$

(we remind that $\gamma=2$ for $q\leq 5/3$) and

$$\Delta_q=\frac{2}{\pi^{1/2}}\left[\frac{q-1}{3-q}\right]^{(3-q)/2(q-1)}\Gamma\left[\frac{3q-5}{2(q-1)}\right]\quad(5/3<q<3)\tag{18}$$

(see Fig. 2). As before, Δ_q essentially characterizes the *width* of the Lévy distribution to which converges, in the $N\to\infty$ limit, the properly scaled distribution $p_q(x,N)$. Summarizing, the width of $p_q(x,N)$ is proportional to $\Delta_q N^{2/\gamma}$. One should be clearly aware that the N dependence of the width is, in principle, very different from its t dependence. Indeed, in contrast to the $q<5/3$ case, for which any standard model is expected to provide $N\propto t$, we expect, for the present case to have $N\propto t^\delta$ with a model-dependent $\delta<1$, in order to take into account the physical time needed for performing very long flights. Consequently, the width we are focusing on would be proportional to $t^{2\delta/\gamma}$. We expect, of course, to be $1<2\delta/\gamma<2$, i.e., superdiffusion.

Let us now address the last point of the present paper, namely, the *escort* [15] distributions $P_q(x)$ appearing, e.g., in Eq. (9). The q-expectation values associated with $p_q(x)$ equal the 1-expectation values associated with

$$P_q(x)\equiv\frac{[p_q(x)]^q}{\int_{-\infty}^{\infty}dx[p_q(x)]^q}\quad(q<3).\tag{19}$$

This distribution is given for $q>1$ by

$$P_q(x)=\frac{1}{\sigma}\left[\frac{q-1}{\pi(3-q)}\right]^{1/2}\frac{\Gamma\left(\frac{q}{q-1}\right)}{\Gamma\left(\frac{q+1}{2(q-1)}\right)}\frac{1}{\left[1+\frac{q-1}{3-q}\frac{x^2}{\sigma^2}\right]^{q/(q-1)}}\tag{20}$$

and, for $q<1$, by

$$P_q(x)=\frac{1}{\sigma}\left[\frac{1-q}{\pi(3-q)}\right]^{1/2}\frac{\Gamma\left(\frac{3-q}{2(1-q)}\right)}{\Gamma\left(\frac{1}{1-q}\right)}\left[1-\frac{1-q}{3-q}\frac{x^2}{\sigma^2}\right]^{q/(1-q)}\tag{21}$$

if $|x|<\sigma[(3-q)/(1-q)]^{1/2}$ and zero otherwise (the cutoff is maintained even if $q\leq 0$). (See Fig. 3.) Since for all $q<3$ (and not only for $q<5/3$), $P_q(x)$ has a *finite* second moment, its N-folded convolution will converge to a Gaussian distribution. More specifically, since their second moment equals σ^2 $(\forall q)$ [see Eq. (9)], this limiting Gaussian distribution is $\propto\exp[-x^2/2N\sigma^2]$. In order to be complete, we can finally focus on the escort distribution associated with the $N\gg 1$ limit of $p_q(x,N)$. For $q<5/3$ this escort distribution is a Gaussian proportional to $\exp\{-[q(5-3q)/2(3-q)]x^2/N\sigma^2\}$. For $5/3<q<3$, the corresponding escort distribution is proportional to $[L_\gamma(x/N^{1/\gamma})]^q$.

Let us conclude by pointing out that the present qualitative results are roughly the same as those exhibited in [4] and quantitatively simpler. For instance, the dimensionless diffusion coefficient Δ_q (see Fig. 2) diverges proportionally to $|q-(5/3)|^{-1}$ on *both* sides of $q=5/3$, in contrast with [4]. Because of this, the *extensive-nonextensive* critical phenomenon occurring in this system is pleasantly analogous with what happens in thermodynamic equilibrium phase transitions. In fact, the analogy is even stronger. Indeed, we can verify that $\lim_{q\to 5/3}[\Delta_{q+0}/\Delta_{q-0}]=1/\pi^{1/2}$, i.e., the Levy-regime diffusion is, at the critical point, *smaller* than the Gaussian regime diffusion. If we consider the Gaussian and Lévy regimes as naturally corresponding to the *disordered* and *ordered* equilibrium phases, respectively, this inequality has precisely the expected sense (as compared to standard critical phenomena results for susceptibility, compressibility, correlation length, etc.). Notice, also, the monotonicity of Δ_q on *both* sides of $q=5/3$ (in [4], only for $q<5/3$, the Δ_q coefficient was monotonic; indeed, although not explicitly shown in [4], for $q\geqslant 5/3$, Δ_q presented a flat minimum at $q\simeq 2.3$). Finally, let us mention a ''paradox,'' which apparently emerges. The above result $\lim_{q\to 5/3}[\Delta_{q+0}/\Delta_{q-0}]<1$ seems to suggest that the system diffuses *less* in the Lévy side than in the Gaussian side, which is of course absurd! What happens is that Δ_q reflects the width of the distribution, say, at midheight, and not at all the crucial weight of the long tails, which are responsible on the Lévy side for the divergence of the second moment. Let us finally address the following question: If a real experiment exhibits superdiffusion with long tails at large distances, to what distribution appearing herein must we compare the experimental results? First of all, let us emphasize that the present calculation only concerns diffusion in which we have reasons to believe that the jumps are *uncorrelated* (otherwise, the present *N*-folded convolution would not describe the macroscopic phenomenon). If this condition is essentially satisfied, we must compare the experiment with the $N\to\infty$ limit of $p_q(x,N)$, which is Gaussian for $q<5/3$ and Lévy distribution for $q>5/3$ [with $\gamma=(3-q)/(q-1)$]. (Notice that, unless $q=1$ or $q=2$, $p_q(x)$ does *not* coincide with the limiting attractors of $p_q(x,N)$. If q is experimentally controlled by parameters like concentrations, temperature, etc., we predict a divergence of the relevant diffusion coefficient when crossing $q=5/3$. If the observed phenomenon concerns only one or a few jumps, then the simple comparison with $p_q(x)$ might be useful, although at this level there are no generic reasons for expecting the results to be universal. The present scheme might be useful for a variety of anomalous diffusive phenomena, like those occurring in Hamiltonian systems including long-range interactions [16] (and even perhaps in some dissipative systems [8,9]), for which the present q statistics seems to be the appropriate framework.

Summarizing our main results: (i) Qualitatively speaking, we have confirmed that the nonextensive statistical mechanics [5] introduced a decade ago, and recently reformulated [11] in terms of *normalized* q-expectation values, unifies the foundations of both Gaussians and Lévy distributions in physical (biological, socioeconomical) systems. Their ubiquity in nature becomes therefore more comprehensive. Also, we have exhibited a strong analogy between standard equilibrium phase transitions and the Gauss (extensive)-Lévy (nonextensive) critical phenomenon. (ii) Quantitatively speaking , we have derived a simple connection, namely, Eq. (17), between the entropic index q and the Lévy index γ (which coincides with the relevant fractal dimension associated with Lévy flights). Also, we have quantified, in Eqs. (16) and (18) and in Fig. 2, the q dependence of the *width* of the properly scaled limiting distributions $p_q(x,N)$ describing the macroscopic ($N\to\infty$) diffusion. In other words, when q increases from $-\infty$ to 3 through $q=5/3$, the distribution monotonically *flattens down* until becoming fully flat at $q=5/3$, and then *deflattens up* until q achieves the uppermost value 3.

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[1] *Lévy Flights and Related Topics in Physics*, edited by M.F. Shlesinger, G.M. Zaslavsky, and U. Frisch, Lecture Notes in Physics (Springer-Verlag, Berlin, 1995).
[2] M.F. Shlesinger and B.D. Hughes, Physica A **109**, 597 (1981); E.W. Montroll and M.F. Shlesinger, J. Stat. Phys. **32**, 209 (1983); in *Nonequilibrium Phenomena II: From Stochastic to Hydrodynamics*, edited by J.L. Lebowitz and E.W. Montroll (North-Holland, Amsterdam, 1984).
[3] P.A. Alemany and D.H. Zanette, Phys. Rev. E **49**, R956 (1994); Phys. Rev. Lett. **75**, 366 (1995); C. Tsallis, A.M.C. de Souza, and R. Maynard, in *Lévy Flights and Related Topics in Physics* (Ref. [1]), p. 267.
[4] C. Tsallis, S.V.F. Levy, A.M.C. Souza, and R. Maynard, Phys. Rev. Lett. **75**, 3589 (1995); **77**, 5442(E) (1996).
[5] C. Tsallis, J. Stat. Phys. **52**, 479 (1988); E.M.F. Curado and C. Tsallis, J. Phys. A **24**, L69 (1991); **24**, 3187(E) (1991); **25**, 1019(E) (1992).
[6] B.M. Boghosian, Phys. Rev. E **53**, 4754 (1996).
[7] V.H. Hamity and D.E. Barraco, Phys. Rev. Lett. **76**, 4664 (1996); D.F. Torres, H. Vucetich, and A. Plastino, *ibid.* **79**, 1588 (1997).
[8] M.L. Lyra and C. Tsallis, Phys. Rev. Lett. **80**, 53 (1998).
[9] F.A. Tamarit, S.A. Cannas, and C. Tsallis, Eur. Phys. J. B **1**, 545 (1998).
[10] A regularly updated bibliography is accessible at http://tsallis.cat.cbpf.br/biblio.htm
[11] C. Tsallis, R.S. Mendes, and A.R. Plastino, Physica A **26**, 534 (1998).
[12] S. Abe, Physica A (to be published).
[13] A.R. Plastino and A. Plastino, Physica A **222**, 347 (1995); C. Tsallis and D.J. Bukman, Phys. Rev. E **54**, R2197 (1996); L. Borland, *ibid.* **57**, 6634 (1998).
[14] A. Araujo and E. Gine, *The Central-Limit Theorem for Real and Banach-Valued Random Variables* (Wiley, New York, 1980), Chap. 2; G. Christoph and W. Wolf, *Convergence Theorems with a Stable Limit Law* (Akademie, Berlin, 1992), p. 17.
[15] C. Beck and F. Schlogl, *Thermodynamics of Chaotic Systems* (Cambridge University Press, Cambridge, 1993).
[16] C. Anteneodo and C. Tsallis, Phys. Rev. Lett. **80**, 5313 (1998); M. Antoni and A. Torcini, Phys. Rev. E **57**, R6233 (1998).